UDC 539.31

ENERGY METHODS IN CERTAIN PROBLEMS OF DAMPING OF SOLUTIONS

PMM Vol.42, № 1, 1978, pp. 136-151

V. L. BERDICHEVSKII
(Moscow)

(Received April 30, 1976)

The energy characteristics of the damping of solutions of variational type systems of equations are introduced and their properties are investigated. Considered as examples are the classical question concerning the damping of solutions in semi-infinite cylindrical and conical domains, as well as the question of the damping of solutions upon receding from the edge of a thin three-dimensional cylindrical domain (plate).

1. Formulation of the problem. Let us consider a domain V with a piecewise-smooth boundary ∂V in an n-dimensional space R_n. The functions $w^a (a, b = 1, \ldots, m)$ of the Cartesian coordinates $x^i (i, j, k, l = 0, 1, \ldots, n-1)$ which are solutions of a system of equations of the form

$$\frac{\partial U}{\partial w^a} - \frac{\partial}{\partial x^i} \frac{\partial U}{\partial w^a_{,i}} = 0 \tag{1.1}$$

are defined in V, where the energy density U is a nonnegative function of x^i, w^a, $w_{,i}^a \equiv \partial w^a / \partial x^i$, which is convex in the set of variables w^a, $w_{,i}^a$, and summation is over the repeated subscripts.

Let a certain domain Ω, on which the quantities $p_a^k n_k = q_a$ are given, be isolated on the surface ∂V ($p_a^k = \partial U / \partial w_{,k}^a$ and n_k are components of the vector normal to ∂V). We give homogeneous boundary conditions on the remaining part S of the surface ∂V. For definiteness, it can be considered that either $p_a^k n_k = 0$ or $w^a = 0$, or the boundary conditions are mixed on S.

We assume that a unique solution of (1.1) with finite energy

$$E = \int_V U\, dv$$

exists for the boundary value data q_a.

If the energy density has a kernel (vanishes on nonzero functions of w^a), and the boundary conditions do not exclude it, then additional conditions which are considered advanced, can be required for uniqueness of the solution.

Let us select the boundary data q_a on Ω from a certain set M. Find the set M for which solutions of (1.1) will decrease with distance from Ω and describe the nature of the decrease. As a rule, it is interesting to investigate the broadest set M, the set $M^\times$ of all boundary data q_a for which the solution has finite energy.

A number of assertions is presented below just for the Laplace equation (taking account of the inhomogeneity and anisotropy), and the system of equations of linear elasticity theory

$$2U = E^{ij}(x^k)\, w_{,i} w_{,j} \tag{1.2}$$

$$2U = E^{ijkl}(x^k)\varepsilon_{ij}\varepsilon_{kl}, \quad 2\varepsilon_{ij} = w_{i,j} + w_{j,i} \tag{1.3}$$

The quadratic forms (1.2) and (1.3) are positive definite. The investigation of the question posed for the system (1.3) is related to the foundation of the Saint—Venant principle [1—6].

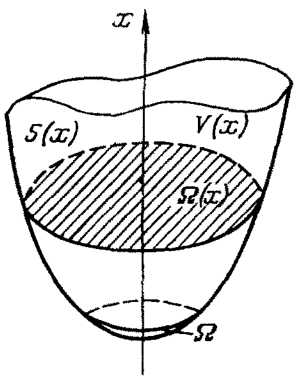

Fig. 1

Let us select a system of coordinates x^i so that the surface Ω would lie on the coordinate surface $x^0 \equiv x = 0$. The coordinates x^i may hence turn out to be curvilinear. Let $\Omega(x)$ denote a section of the domain V by the coordinate planes $x = \text{const}$, $\Omega(0) = \Omega$. The section $\Omega(x)$ divides the domain V and its boundary ∂V into two parts. Those parts of V and ∂V which do not contain Ω are denoted by $V(x)$ and $S(x)$, $V(0) = V$ and $S(0) = S$ (see Fig. 1).

Following [5], we take the energy of the solution $E(x)$ in the domain $V(x)$

$$E(x) = \int_{V(x)} U\, dv$$

as the main characteristic of the solution.

The function $E(x)$ is decreasing because of the positivity of U. Characteristics of the damping of $E(x)$, independently of the selection of boundary data from M are presented and investigated below. An elementary proof is given of the exponential damping of the energy in semi-infinite cylindrical domains and of the power-law damping in conical domains. A class of systems is described, for which exponential energy damping holds independently of M and the geometry of the domain V. The relation between the energy characteristics introduced and the eigennumbers is indicated for the systems (1.2) and (1.3) in the case of a semi-infinite cylindrical domain. Certain sets of M for which the solutions of the systems (1.2) and (1.3) damp out exponentially with removal from the edge of a thin three-dimensional cylindrical domain, are indicated. In this connection, a method is given for constructing the approximate solutions in thin domains with an error less than any power of the relative thickness.

Let us note that a theorem of the mean is proved for the systems (1.2) and (1.3) in the case of homogeneity and isotropy [7]: the square of the stress at the center of a sphere does not exceed the energy of the sphere

$$p_a{}^i p_a{}^i \leqslant c \int U\, dv \tag{1.4}$$

Hence, pointwise estimates [5] result from the energy estimates.

2. The function γ. Let us examine the following boundary value problem for the system of equations (1.1) in the domain $V(x)$: The boundary conditions on $S(x)$ are the same as in the initial problem while $p_a{}^k n_k = q_a$ are given on $\Omega(x)$. The boundary data on $\Omega(x)$ are selected from a certain set $M(x)$. Hence: 1) the quantities $p_a{}^k n_k$ on $\Omega(x)$, where $p_a{}^k$ is the solution of the initial problem in V, enter into $M(x)$ for any boundary data on Ω from M; 2) $M(0) \equiv M$; 3) a solution with finite energy exists in the domain $V(x)$ for any q_a from $M(x)$.

In general, for any boundary data from $M(x)$ a solution of the problem posed can be found and the appropriate "surface" and volume energies

$$E_\Omega(q_a) = \int\limits_{\Omega(x)} U\,d\omega, \quad E_V(q_a) = \int\limits_{V(x)} U\,dv$$

can be calculated.

The argument q_a emphasizes the dependence on the selection of the boundary data in $\Omega(x)$. It is not excluded that the surface energy becomes infinite for certain q_a.

Let us define the quantity γ by the relationship [6]

$$\gamma = \inf_{q_a \in M(x)} \frac{E_\Omega(q_a)}{E_V(q_a)} \tag{2.1}$$

According to (2.1) the quantity γ has the dimensionality $(\text{length})^{-1}$ and is a function of x. The function $\gamma(x)$ depends on the geometry of the domain V, the coefficients $E_1, \ldots, E_s$ in the energy U, the selection of $\Omega(x)$ and $M(x)$ and is independent of the specific boundary data q_a displayed in $\Omega(x)$.

Let $E(x)$ be the energy of any solution corresponding to the boundary data q_a from M. Let us show that the estimate

$$E(x) \leqslant E_0 \exp\Big(-\int\limits_0^x \gamma(x)\,dx\Big), \quad E_0 \equiv E(0) \tag{2.2}$$

holds for $E(x)$. Indeed, from (2.1)

$$\gamma(x) E(x) \leqslant \int\limits_\Omega U\,d\omega = -\frac{dE}{dx} \tag{2.3}$$

from which (2.2) results.

Notes. 1) The estimate (2.2) is understandably meaningful only in case $\gamma(x) \not\equiv 0$. This latter is proved in [6] for linearly elastic bodies; the proof for other linear systems is analogous (*).

2) It can be seen that the estimate (2.2) remains valid if the domain of variation of the coordinates $x^1, \ldots, x^{n-1}$ varies by a jump, as holds for a cylindrical rod comprised of two rods with different cross-section.

3. **On the calculation of γ.** The definition (2.1) does not appear constructive. Nevertheless, the dependence of γ on x is explicitly found successfully

---

*) After the paper had been sent to press, a number of papers [8—11] appeared in which inequalities were obtained which, essentially, represent some lower bounds for γ for one elliptic equation and for the equations of elasticity theory. These estimates permitted an indication of the uniqueness class of the solutions of boundary value problems in unbounded domains. Analogous questions were investigated for parabolic equations in [12.].

in a number of cases characterized by the presence of a sufficiently rich symmetry group in the problem. Let us illustrate the above by examples.

1°. **Semi-infinite cylinder with a free surface.** Let the domain V be a semi-infinite cylinder with a generator along the x axis, $x \geqslant 0$, Ω is the base of the cylinder in the $x = 0$ plane, $p_a^k n_k = 0$ on S, $M = M(x) = M^\times$ and U explicitly independent of x. It follows from the definition (2.1) that the function $\gamma(x)$ is invariant relative to translations along the x axis, and therefore, is a constant. Therefore

$$E(x) \leqslant E(0)\, e^{-\gamma x} \tag{3.1}$$

Furthermore, let the energy density U be explicitly independent of the remaining coordinates. Then γ is a function of the diameter h of the domain Ω, the shape of Ω and the coefficients $E_1, \ldots, E_s$ in U. If $E_1, \ldots, E_s$ have a dimensionality independent of the length, then by the π-theorem [13]

$$\gamma = \gamma^* / h \tag{3.2}$$

where γ^* is a function of the cross-sectional shape and dimensionless quantities formed from $E_1, \ldots, E_s$. Formula (3.2) is valid particularly for systems with the energies (1.2) and (1.3).

The relationships (3.1), (3.2) and (1.4) show that solutions with finite energies in a semi-infinite cylinder are always of boundary layer nature.

2°. **Self-equilibrated boundary data and finiteness of the energy.** In cases when the energy density has an invariance group, the condition of finiteness of the energy in problems with a free boundary imposes a number of constraints on the boundary data. Let us find these constraints.

Let U have an r-parameter invariance group: $U(x, w^a, w_{,i}^a) = U(x^i, w'^a, w_{,i}'^a)$, where $w'^a = w^a + \psi_s^a(x^k)\,\omega^s$, $\omega^1, \ldots, \omega^r$ are group parameters, and ψ_s^a are known functions of x^k. Then there are r integrals

$$\langle \psi_s^a \partial U / \partial w_{,x}^a \rangle = P_s = \text{const} \tag{3.3}$$

in the problem. Here $\langle \cdot \rangle$ is the integral over the cross section. Indeed, there results from the invariance of U that the identity $\psi_s^a \partial U / \partial w^a + \psi_{s,k}^a \partial U / \partial w_{,k}^a \equiv 0$ is valid for any w^a. Multiplying (1.1) by ψ_s^a, integrating over the cross section and using the identity noted and the boundary conditions on S, we obtain $\partial \langle \psi_s^a \partial U / \partial w_{,x}^a \rangle / \partial x = 0$, from which (3.3) follows.

Let us assume that U possesses the following property

$$\text{from } U \to 0 \Rightarrow \partial U / \partial w_{,x}^a \to 0 \tag{3.4}$$

The condition of finiteness of the energy means that $U \to 0$ as $x \to \infty$. It follows from (3.4) that $\partial U / \partial w_{,x}^a \to 0$ as $x \to \infty$. As a rule, this permits evaluation of the constant P_s in (3.3). From (3.3) and the boundary conditions on Ω we obtain the constraints on the edge data

$$\langle \psi_s^a q_a \rangle = -P_s$$

Let us illustrate the above by an example of the system (1.2). The boundary conditions for $x = 0$ have the form $-\partial U / \partial w_{,x} = q$. The energy density satisfies the condition (3,4) and is invariant relative to translations in w : $w' = w + \psi\omega$, $\psi \equiv 1$. Letting x tend to infinity, we find from (3.3) that $P = 0$, and therefore

$$\langle q \rangle = 0 \tag{3.5}$$

If w is the temperature (potential incompressible fluid flow), then E has the meaning of dissipation (kinetic energy), and (3.5) is the condition that the total heat flux (fluid discharge) is zero.

The energy density of an elastic solid has a six-parameter invariance group (the group of body motions as a solid). In conformity with this, the condition of finiteness of the energy results in six constraints: the conditions that the resultant and total moment of the forces applied on Ω will vanish.

We shall designate the edge data satisfying conditions of the type (3.5) as self-equilibrated. The requirement of self-equilibration of the edge data, as well as their certain smoothness (for the system (1.2) and (1.3) $q_a \in L_2$), is sufficient for the existence of a solution with finite energy.

3°. Semi-infinite cylinder with "clamped" section of the side surface. Let us divide the boundary Γ of the domain Ω into two parts: Γ_p and Γ_w and let us set $S_p = \Gamma_p \times [0, \infty]$, $S_w = \Gamma_w \times [0, \infty]$.

We consider that $p_a{}^k n_k = 0$ on S_p and $w^a = 0$ on S_w. Exactly as in Sect. 1°, we arrive at (3.1) and (3.2). However, the condition of finiteness of the energy is now not related to the condition of self-equilibration of the edge data since (3.3) does not hold. The stresses in an elastic body (the heat flux) will damp exponentially even in the case when the total forces and moments on the endface (total heat flux) differ from zero.

4°. Semi-infinite cylinder with periodic boundary conditions. Let periodic boundary conditions be given on the half-space boundary. Isolating an elementary periodically repeated cell on the boundary, we arrive at the problem of a semi-infinite cylinder with cross-section in the form of a parallelogram on whose opposite edges the periodicity conditions of w^a and $p_a{}^k$ are imposed. Repeating the discussion in Sect. 2 and 3. 1° word for word, we obtain (3.1) and (3.2)

5°. On the formulation of the Saint—Venant principle for cylindrical elastic rods. Examples 3°, 4° show that self-equilibration of the load cannot be a necessary condition for damping the stresses. The general criterion is a condition for finiteness of the energy: in order for the stresses to damp exponentially with distance from the endface of a semi-infinite cylindrical elastic rod homogeneous in direction of the axis, on whose side surface either the surface forces or displacements vanish or the periodicity conditions are satisfied, it is necessary and sufficient that the energy of the rod be finite. The rate of energy damping γ for a completely homogeneous rod has the form $\gamma = \gamma^* / h$, where γ^* depends only on the shape of the cross-section and the elastic moduli, and h is the diameter of the cross-section.

The necessity of the assertion is evident, the sufficiency is proved in Sects. 1°-4°. For a rod of finite length l the condition of finiteness of the energy in the previous formulation must be replaced by the requirement: the energy will remain bounded as $l \to \infty$.

Later only the systems (1.2) and (1.3) are examined to the end of Sect. 3.

6°. **Expanding cone.** Let us take the domain Ω in the $x = 0$ plane, and draw lines through points of the boundary Ω and a point lying on the negative part of the x-axis at a distance l from the origin (see Fig. 2).

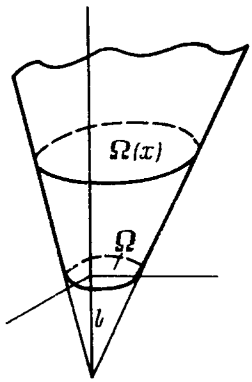

Fig. 2

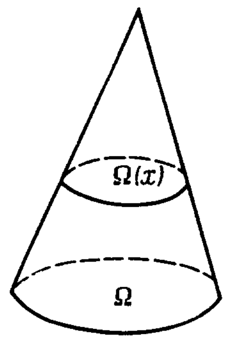

Fig. 3

We consider the expanding cone V to be part of the half-space $x \geqslant 0$ lying within the conical surface being obtained. We consider the conical part of the boundary free. Let us set $M(x) \equiv M$. Two methods of selecting the set M are hence most interesting: $M = M^{\times}$ and $M = M^{\times\times}$, where $M^{\times\times}$ is the set of self-equilibrated edge data. In contrast to the problem of Sect. 1°, these sets are distinct (for example, quantities of the type $\langle \partial U / \partial w_{,x} \rangle$ are independent of x for the system (1.2), however, they do not vanish absolutely since the domain of integration increases together with the decrease in $\partial U / \partial w_{,x}$ as $x \to \infty$).

First let $M(x) = M^{\times\times}$. It follows from the definition (2.1) that γ depends on $x + l$, the shape of Ω, the solid angle of the cone α and the parameters $E_1, \ldots, E_s$. According to the π-theorem

$$\gamma = \gamma^* / (x + l) \tag{3.6}$$

where γ^* has the same meaning as in (3.2). Introducing $\gamma(0) = \gamma^* / l$, then (3.6) can be rewritten as $\gamma = \gamma(0)\,(1 + x / l)^{-1}$. The estimate (2.2) becomes

$$E(x) \leqslant E_0 (1 + x / l)^{-\gamma(0) l} \tag{3.7}$$

Therefore, the energy decreases in the case of a cone, at least, according to a power law with the exponent $\gamma(0)\, l$. It is natural to expect that $\gamma(0)$ depends continuously on the cone shape and tends to the appropriate value of the constant γ for a cylinder $-\ \gamma_0$ upon degeneration of the cone into a cylinder (as $l \to \infty$). Hence, the power law of the decrease (3.7) goes over into the exponential law (3.1) for edge data with an identical value of the total energy E_0.

If $M(x) = M^{\times}$ then reasoning analogously, we arrive at (3.7), however, the passage to the limit from a cone to a cylinder becomes impossible since E_0 hence generally becomes infinite.

7°. **Contracting cone.** Now, let us set the cone apex at the point $x = l$ of the positive part of the x-axis, hence the domain will be bounded by a cone and a section of the plane Ω (Fig. 3) and have a finite volume. Let the conical part of the boundary V be free. In the case under consideration γ is a function of $(l-x)$, the shape of Ω, the solid angle of the cone, and the parameters $E_1, E_2, \ldots, E_s$. According to the π-theorem, $\gamma = \gamma^* / (l-x) = \gamma(0)(1 - x/l)^{-1}$. The estimate (2.2) yields

$$E(x) \leqslant E(0)(1 - x/l)^{\gamma(0)l} \tag{3.8}$$

As $l \to \infty$, (3.8) goes over into (3.1). Formulas (3.1), (3.7) and (3.8) permit tracing how the nature of the damping depends on the body shape. As the expanding cone contracts (the "approach" of the free boundary to the loading site) the power-law damping speeds up until it passes over into an exponential law. As the cylinder "collapses" further, the damping becomes still more rapid in the contracting cone $((1 - x/l)^{\gamma(0)l} < e^{-\gamma(0)x})$.

4. **Systems with autonomous damping.** The nature of the energy damping in the systems (1.2) and (1.3) depends essentially on the geometry of the domain. A class of systems with damping independent of the shape of the domain boundary can be extracted. Let us present one sufficient condition for autonomy of the damping. Let $\Omega = \partial V$, M be arbitrary, $M(x)$ be a set of boundary conditions on $\Omega(x)$ induced by solutions of the problem with edge data from M, the energy density be such that for any $w^a, w_{,i}^a$ and any unit vector n^k the inequality

$$\alpha \,|w^a n^k \partial U / \partial w^a_{,k}| \leqslant U(x^k, w^a, w^a_{,i}), \quad \alpha = \text{const} \tag{4.1}$$

holds

Then by virtue of (1.1), (4.1) and the inequality $U \leqslant w^a \partial U / \partial w^a + w_{,k}^a \, \partial U / \partial w_{,k}^a$ which follows from the convexity of U, we have

$$E(x) \leqslant \int\limits_{V(x)} \left(w^a \frac{\partial U}{\partial w^a} + w^a_{,k} \frac{\partial U}{\partial w^a_{,k}} \right) dv = \int\limits_{\Omega(x)} w^a \frac{\partial U}{\partial w^a_{,k}} n_k \, d\omega \leqslant \alpha^{-1} \int\limits_{\Omega(x)} U \, d\omega \tag{4.2}$$

Therefore, for any M we will have $\alpha \leqslant \gamma$ and $E(x) = E(0) \exp(-\alpha x)$. For linear systems (4.2) can be strengthened by setting the equality sign instead of the inequality and the coefficient $^1/_2$ in front of the integral in first relationship in (4.2). Hence $2\alpha \leqslant \gamma$ and

$$E(x) \leqslant E(0) \exp(-2\alpha x) \tag{4.3}$$

The inequality (4.1) can hold only in cases when the energy density depends explicitly on w^a. It is satisfied, for instance, for systems with energy of the form

$$U = U_1(w^a) + U_2(w^a_{,i}), \quad w^a w_a \leqslant \text{const}\, U_1 \tag{4.4}$$

$$\frac{\partial U}{\partial w^a_{,i}} \frac{\partial U}{\partial w^a_{,i}} \leqslant \text{const}\, U_2$$

Plates on an elastic foundation, in particular, belong to systems of the type (4.4).

In the case of one equation with the energy density $2U = w^2 + h^2 w_{,i} w_{,i}$ (h is a small parameter), $\alpha = h^{-1}$ and (4.3) together with a formula of the type (1.4) yield an elementary proof that the solution is in the nature of a boundary layer.

5. **Some estimates of γ.** Let us examine the system of equations (1.1) in a cylindrical domain of finite height l, $0 \leqslant x \leqslant l$. Let us consider that U can depend explicitly on all the coordinates x^i and be independent of w^a. We can write for the energy $E(x)$

$$E(x) = \int_x^l \langle U \rangle\, dx \leqslant \int_x^l \left\langle w^a_{,i} \frac{\partial U}{\partial w^a_{,i}} \right\rangle dx = - \int_{\Omega(x)} w^a \frac{\partial U}{\partial w^a_{,x}}\, d\omega \tag{5.1}$$

If the inequality

$$\alpha(x) \left| \langle w^a \partial U / \partial w^a_{,x} \rangle \right| \leqslant \langle U \rangle \tag{5.2}$$

holds, then we obtain $E(x) \leqslant \alpha^{-1} \langle U \rangle$ from (5.1), and therefore the estimate of γ: $\alpha(x) \leqslant \gamma(x)$. Let us indicate the sufficient conditions for compliance with the inequality (5.2). Let U satisfy the constraints

$$a \frac{\partial U}{\partial w^c_{,x}} \frac{\partial U}{\partial w^c_{,x}} \leqslant U(x^i, w^c_{,i}), \quad b w^a_{,\alpha} w_{a,\alpha} \leqslant U(x^i, w^a_{,i}) \tag{5.3}$$

If $w^a = 0$ on part of the boundary of Ω by virtue of the boundary conditions, then as is known, the inequality [14]

$$\lambda^2 \langle w^a w_a \rangle \leqslant \langle w^a_{,\alpha} w_{a,\alpha} \rangle \tag{5.4}$$

holds. We have from (5.3) and (5.4)

$$\begin{gathered} \left| \langle w^a \partial U / \partial w^a_{,x} \rangle \right| \leqslant \left(\langle (\partial U / \partial w^a_{,x}) \partial U / \partial w^a_{,x} \rangle \langle w^a w_a \rangle \right)^{1/2} \leqslant \\ \left(a^{-1} \langle U \rangle \lambda^{-1/2} \langle w^a_{,\alpha} w_{a,\alpha} \rangle \right)^{1/2} \leqslant a^{-1/2} b^{-1/2} \lambda^{-1} \langle U \rangle \end{gathered} \tag{5.5}$$

Therefore, the inequality (5.2) with the constant $\alpha = (ab)^{1/2}\lambda$ is valid.

Now, let the boundary be free. If the edge data are self-equilibrated $\langle q_a \rangle = 0$, then $\langle \partial U / \partial w_{,x}{}^a \rangle = 0$. Hence

$$\left\langle w^a \frac{\partial U}{\partial w^a_{,x}} \right\rangle = \left\langle (w^a - \langle w^a \rangle) \frac{\partial U}{\partial w^a_{,x}} \right\rangle \tag{5.6}$$

There remains to use the inequality

$$\lambda^2 \langle (w^a - \langle w^a \rangle)(w_a - \langle w_a \rangle) \rangle \leqslant \langle w^a_{,\alpha} w_{a,\alpha} \rangle \tag{5.7}$$

instead of (5.4). The values of the constant λ in (5.4) and (5.7) are understandably distinct. Analogously to (5.5), we obtain $\alpha = (ab)^{1/2}\lambda$ from (5.6), (5.3) and (5.7).

The functions a and b in the expression for α are related only to the characteristics of the system, and λ to the geometry of the cross-section. Evidently λ has the form $\lambda = \lambda^*/h$, where λ^* is a function of just the shape of the cross-section, hence $\alpha = \alpha^*/h$, α^* is a function of the shape of the cross-section and the system characteristics.

The case of a curvilinear cylinder and other non-cylindrical domains reduces to the coordinate transformation considered. Appropriate estimates will be satisfactory if the product ab is not too small.

6. The constant γ for a semi-infinite cylinder and eigenvalues. Only linear systems will be considered in the rest of the paper (U is a positive quadratic form of the variables w^a and $w_{,i}{}^a$). The solution of problems 1° and 3° of Sect. 3 for linear systems in a semi-infinite cylinder can be constructed in the form of series in functions of the form $e^{-\varkappa x}u^a(x^\alpha)$ (the Greek superscripts α, β, γ, ... run through the values $1, 2, \ldots, n-1$ and correspond to coordinates in the plane $x = 0$). Hence $u^a(x^\alpha)$ and $\varkappa$ are the eigenfunctions and the eigenvalue of some eigenvalue problems. The nature of the damping of the solutions (for $q_a \in M^\times$) is determined by the least eigenvalue δ. The question arises as to how δ and γ are related.

Let us examine it first in an example of the system (1.2). Let us consider the energy density invariant for reflections relative to a plane perpendicular to the cylinder axis. This means that products of the form $w_{,\alpha}\, w_{,x}$ do not enter into (1.2), i.e.,

$$2U = E^{\alpha\beta}(x^\gamma)\, w_{,\alpha} w_{,\beta} + E(x^\alpha)\, w_{,x}^2 \tag{6.1}$$

A self-adjoint problem in the domain Ω is obtained to determine $u(x^\alpha)$ and $\varkappa$ (ν_α are components of the normal vector to the boundary Γ of the domain Ω)

$$(E^{\alpha\beta}u_{,\beta})_{,\alpha} + E\varkappa^2 u = 0, \quad E^{\alpha\beta}u_{,\beta}\nu_\alpha|_\Gamma = 0 \tag{6.2}$$

The appropriate eigenvalues are real, separated from zero, and their set is countable. The eigenfunctions corresponding to different eigenvalues are orthogonal relative to the scalar products $\langle E^{\alpha\beta}u_{,\alpha}u_{,\beta}\rangle$ and $\langle Euv\rangle$. Expanding the solution in a series of eigenfunctions, we can write for γ (the coefficients of the expansion include factors in the eigenfunctions)

$$\gamma - \inf_{u_{(k)}} \sum_k U_{(k)} \Big/ \sum_k U_{(k)}(2\varkappa_{(k)})^{-1} \tag{6.3}$$

$$2U_{(k)} = \langle E^{\alpha\beta}u_{(k),\alpha}u_{(k),\beta}\rangle + \varkappa_{(k)}^2\langle Eu_{(k)}^2\rangle$$

It hence follows that $\gamma = 2\delta$.

The problem of eigenfunctions for an elastic body is non-self-adjoint in contrast to the preceding problem, hence the eigennumbers are complex. It can be shown that $\gamma \leqslant 2\delta$, where δ is the minimum value of the positive real parts of the eigenvalues. If the eigenfunction u^i corresponding to the eigenvalue with real part δ is orthogonal on energy to the remaining eigenfunctions, then $\gamma = 2\delta$, otherwise $\gamma < 2\delta$.

7. The function $\beta(x)$. In the self-adjoint problem examined in Sect. 6, γ differs only by a factor from the least eigenvalues, and hence, possesses a number of remarkable properties resulting from the Rayleigh variational formula for the least eigenvalue; in particular, γ is increased upon the imposition of constraints. The question arises: is this property not common for all problems ?. Without answering this question, let us however show that γ can be estimated by a certain quantity β possessing a number of partial Rayleigh properties as lower bound.

Let us assume that the energy density U satisfies the condition

$$\frac{1}{\mu}\left(\frac{\partial U}{\partial w^a_{,i}} n^i \frac{\partial U}{\partial w^a_{,k}} n^k\right) \leqslant U(x^i, w^a, w^a_{,i}) \tag{7.1}$$

where μ is independent of w^a, $w_{,i}{}^a$, n^i are components of the unit vector on Ω. We define $\beta(x)$ by the formula

$$\beta(x) = \inf_{q_a \in M(x)} \frac{\langle \mu^{-1} q_a q^a \rangle}{E_V(q_a)} \tag{7.2}$$

In contrast to (2.1), the number of the ratio in (7.2) is determined by the boundary conditions on Ω, and therefore, the solution of the boundary value problem enters into this ratio only in terms of the volume energy. We have from (7.1), (7.2) and (2.1)

$$\beta(x) \leqslant \gamma(x) \tag{7.3}$$

The function $\beta(x)$ possesses the following properties (the term "less" is used throughout in the sense of "less than or equal to"):

1) If the characteristics of the system with energy density U_1 are less than the characteristics of a system with energy density U_2 (in the sense that $U_1(x^i, w^a, w_{,i}{}^a) \leqslant U_2(x^i, w^a, w_{,i}{}^a)$ for any x^i, w^a, $w_{,i}{}^a$ and $\mu_1^{-1} \leqslant \mu_2^{-1}$), then the corresponding values of the function $\beta_1(x)$ of the first system are less than the corresponding values of the function $\beta_2(x)$ of the second system.

2) The function $\beta(x)$ grows with the expansion of the part of the boundary on which the values $w^a = 0$ are given.

3) Let the geometry of the domain V allow to set of periodic boundary conditions on S (for instance, V is a cylinder with cross-section in the form of a parallelepiped, and Ω is the cylinder endface). Then the function β_1 evaluated for the free surface S ($p_a{}^k n_k = 0$ on S), is less than $\beta_2(x)$ evaluated for the periodic boundary conditions.

4) Let us examine the partition of the domain $V(x)$ by a surface $R(x)$ into two subdomains $V_1(x)$ and $V_2(x)$, and let $\Omega_1(x)$ and $\Omega_2(x)$ denote the parts of $\Omega(x)$ lying on the boundaries of $V_1(x)$ and $V_2(x)$, while $M_1(x)$ and $M_2(x)$ are sets of boundary conditions induced by the set $M(x)$ on $\Omega_1(x)$ and $\Omega_2(x)$. We define $\beta_1(x)$ by (7.2), in which $V(x)$ is understood to be $V_1(x)$ while $\Omega(x)$ is $\Omega_1(x)$, and the set $M_R(x)$ of boundary conditions of the form $p_a{}^\kappa n_k = r_a$ is taken on $R(x)$ so that the problem for the domain $V_1(x)$ would be solvable. Then

$\beta_2(x)$ is defined analogously, where the boundary conditions $p_a{}^k n_k = -r_a$ are taken on R. Then $\beta(x) \geqslant \min(\beta_1(x), \beta_2(x))$.

5) Let $p_a{}^k n_k = 0$ on S. We examine the domain $V_1(x)$ imbedded in $V(x)$ and such that the boundary of $V_1(x)$ contains $\Omega(x)$. We define the quantity $\beta_1(x)$ by the relationship (7.2) in which E_V is understood to be the energy of the solution of the boundary value problem for the domain $V_1(x)$ with homogeneous boundary conditions $p_a{}^k n_k = 0$ outside $\Omega(x)$. Then $\beta_1(x) \leqslant \beta(x)$.

The properties presented for $\beta(x)$ are due to the corresponding properties of the energy (see the appendix in [6]).

By using the property 1), it is possible to estimate $\beta_2(x)$ in terms of the function $\beta_1(x)$ of a simpler system (for example, U_1 can be taken as the energy density of an isotropic body with the Lamé coefficient $\lambda = 0$ in problems of the theory of anisotropic elasticity). The properties 2) and 3) show that $\beta(x)$ increases upon the imposition of additional kinematic constraints. On the basis of properties 4) and 5) there is the possibility of estimating $\beta(x)$ in terms of values of $\beta_1(x)$ for bodies with simple geometric shape. (This method was used in [6]). In particular, the proof of the positivity of γ for a body of arbitrary shape can be based on the property 5) and (7.3); it is sufficient to show that the quantity β is not zero, for example, for a contracting cone with a sufficiently blunt angle at the apex.

For a cone of height h resting on a domain of diameter h, evidently $\beta = \beta^* / h$. Hence, for example, for a homogeneous elastic curvilinear rod of constant cross-section, we obtain the estimate $\beta^* / h \leqslant \gamma$ by virtue of (7.3) and the property 5) by inscribing a cone resting on the cross-section, which will assure exponential damping of the solution with removal from the endface, with at least the velocity β^* / h.

It is interesting to clarify how rough is the estimate of γ in terms of β (7.3). We show that in problem 1° of Sect. 3, β is half γ for the system (1.2). We have $\mu = 2E(x^\alpha)$ from (7.1). Writing down a formula analogous to (6.3) for γ for β and taking into account that $\langle E^{\alpha\beta} u_{(k),\alpha} u_{(k),\beta} \rangle = \varkappa_{(k)}{}^2 \langle E u_{(k)}{}^2 \rangle$, we obtain $\beta = \delta$.

It follows from the formulation of the eigenvalue problem (6.2) that δ agrees with the least frequency ω of the free natural vibrations of a system with the potential energy (6.1) and the kinetic energy $\frac{1}{2} E u_{,t}{}^2$. The relation $\beta = \omega$ together with the definition (7.2) written for a cylindrical domain results in a new variational formula for the least eigenfrequency.

8. On exponential damping of the solution at the edge of a thin cylindrical domain. Let V be a cylinder with height h and cross-sectional diameter L, $h \ll L$. In what boundary value problems posed for a linear system of the form (1.1) will the solution decrease with distance from the edge (cylindrical part of the boundary), at least exponentially, with the exponent of the form c/h, where c is a number ?.

Let us extract a $(n-1)$-dimensional strip in V Fig. 4). This strip is a cylinder with cross-sectional diameter considerably less than the length. The exponential damping conditions in such cylinders have been formulated in Sect. 3. It is natural to assume that upon complying with these conditions on each strip cut from V, the solution will damp exponentially as a whole. Let us describe the class of systems for which the assumption expressed turns out to be valid.

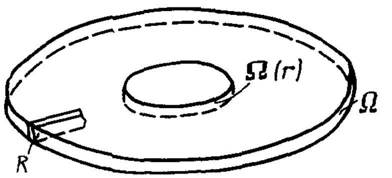

Fig. 4

Let us dispose the cylinder relative to the coordinate axes exactly as in Sect. 6, $-h/2 \leqslant x \leqslant h/2$, let T_+ and T_- denote the plane faces of the cylinder, and T and Γ a section of the cylinder by the plane $x = 0$ and its boundary. The domain T is bounded and simply-connected for simplicity. In the terminology defined above $\Omega = \Gamma \times [-h/2, h/2]$, $S = T_+ + T_-$. The edge conditions on T_+, T_- are taken so that the "work of the external forces" would be zero at each point of T_+, T_-, i.e.,

$$w^a \partial U / \partial w^a_{,x} = 0 \tag{8.1}$$

Let us introduce a one-parameter family of contours $\Gamma(r)$ and the corresponding cylindrical surfaces $\Omega(r) = \Gamma(r) \times [-h/2, h/2]$, as is shown in Fig. 4. Using (8.1) and (1.1), the solution for the energy in the domain $V(r)$ can be written as

$$E(r) = \frac{1}{2} \int_{\Gamma(r)} \left\langle w^a \nu_a \frac{\partial U}{\partial w^a_{,\alpha}} \right\rangle ds \tag{8.2}$$

Here and henceforth $\langle \cdot \rangle$ is the integral with respect to x between the limits $-h/2$, $h/2$, and ν_α are components of the external unit normal vector. That the decrease in $E(r)$ is exponential can be proved if the inequality

$$E(r) \leqslant c^{-1} h \int_\Gamma \langle U \rangle ds, \quad c = \text{const} \tag{8.3}$$

is successfully deduced.

We then obtain a lower bound for γ from (2.1): $c/h \leqslant \gamma$, which indeed assures the required nature of the damping.

Let us enlarge the integrand in (8.2) by using the Cauchy – Buniakowski inequality

$$\langle w^a \nu_\alpha \partial U / \partial w^a_{,\alpha} \rangle \leqslant (\langle \nu^\alpha \nu^\beta \, (\partial U / \partial w^a_{,\alpha}) \partial U / \partial w_{a,\alpha} \rangle \langle w^b w_b \rangle)^{1/2} \tag{8.4}$$

The first factor in (8.4) is estimated in terms of $\langle U \rangle$ if the system possesses the property (7.1). An estimate for the second factor in (8.4) in terms of $\langle U \rangle$ is possible, for instance, for systems satisfying the condition (4.4). Furthermore, let us examine the less trivial case when U is explicitly independent of w^a and the estimate of $\langle w_a w^a \rangle$ in terms of $\langle U \rangle$ is performed by using the inequality

$$\lambda^2 \langle w^a w_a \rangle \leqslant \left\langle \frac{dw^a}{dx} \frac{dw_a}{dx} \right\rangle, \quad \lambda = \frac{\lambda^*}{h} \tag{8.5}$$

as in Sect. 5. The inequality (8.5) is valid if w^a satisfy the constraints, except the translation in w^a by a constant. For example, $\langle w^a \rangle = 0$ or $w^a(h/2) = 0$ or it is known that $w^a(x, x^\alpha)$ are odd functions, etc. Therefore, for systems satisfying (7.1), as well as the condition

$$b \frac{\partial w^a}{\partial x} \frac{\partial w_a}{\partial x} \leqslant U \tag{8.6}$$

the inequalities (8.4)—(8.6) result in the required inequality (8.3). This means is successfully realized for the system (6.1), in particular.

Exponential damping of the energy holds in the following boundary value problems for the system (6.1).

1°. The domain Γ is divided into two parts Γ_p and Γ_w, $w_{,x} = 0$ on T_+ and T_- ; $E^{\alpha\beta} w_{,\beta} \nu_\alpha = q(x^\alpha, x)$ on $\Gamma_p \times [-h/2, h/2]$; $w = S(x^\alpha, x)$ on $\Gamma_w \times [-h/2, h/2]$, where

$$\langle q^\alpha \rangle = 0 \ \text{ on } \Gamma_p, \quad \langle s^\alpha \rangle = 0 \ \text{ on } \Gamma_w \tag{8.7}$$

2°. The boundary conditions are arbitrary on Ω , $w = 0$ on T_+ and $ww_{,x} = 0$ on T_- .

Proof. Let us integrate (1.1) written for the system (6.1) with respect to x within the limits $-h/2$, $h/2$. Taking account of the boundary conditions, we obtain for the boundary value problem 1°

$$(E^{\alpha\beta} \langle w \rangle_{,\beta})_{,\alpha} = 0 \tag{8.8}$$

The solution of the boundary value problem (8.8) for $\Gamma_w \neq \varnothing$ is unique and has the form

$$\langle w \rangle = 0 \text{ in } T \tag{8.9}$$

If $\Gamma_w = \varnothing$, then it follows from (8.8) that $\langle w \rangle = \text{const}$. Since the solution of the initial problem in V is determined to the accuracy of a constant for $\Gamma_w = \varnothing$, this constant can be selected so that (8.9) would be satisfied. Now (8.5) holds with the best constant $\lambda^* = \pi$, by virtue of (8.9) while the best constant in the inequality (8.6) is $b = E/2$. We find from (8.4)—(8.6)

$$\langle w\nu_\alpha E^{\alpha\beta} w_{,\beta} \rangle \leqslant (\langle E^{\sigma\beta} w_{,\beta} E_\sigma{}^\alpha w_{,\alpha} \rangle \langle w^2 \rangle)^{1/2} \leqslant h\pi^{-1} (\langle E^{\alpha\beta} E_{\max} w_{,\alpha} w_{,\beta} \rangle \ \langle w_{,x}^2 \rangle)^{1/2} \leqslant \tag{8.10}$$
$$h\pi^{-1}\nu^{-1} \langle U \rangle, \quad \nu = \min_{x^\alpha \in T} (E(x^\alpha) / E_{\max}(x^\alpha))^{1/2}$$

where $E_{\max}$ is the maximum eigenvalue of the tensor $E^{\alpha\beta}$. Substituting (8.10) into (8.2) results in (8.3) with the constant $c = 2\pi\nu$. The proof for the boundary value problem 2° requires no derivation of an inequality of the type (8.9), since the inequality (8.5) turns out to be valid directly because of the boundary condition on T_+ with the best constant $\lambda^* = \pi/2$. In this case we obtain $c = \pi\nu$ for the damping constant c .

Let us emphasize that the energy damps exponentially with a velocity $2\pi\nu/h$ in boundary value problem 1° and velocity $\pi\nu/h$ in boundary value problem 2° in each $(n-1)$-dimensional strip R with induced boundary conditions for a homogeneous medium. This verifies the assumption expressed at the beginning of Sect. 8, and shows that the values obtained for the constants c are best. For inhomogeneous anisotropic media, the estimate obtained can be strengthened since the inequality (8.3) remains valid if the constant c is related to ν by the previous formulas and ν is understood to be a function of r

$$\nu(r) = \min_{x^\alpha \in \Gamma(r)} \left(E(x^\alpha) / E_{\max}(x^\alpha)\right)^{1/2}$$

We hence find for $E(r)$ in problem 1°

$$E(r) \leqslant E(0) \exp\left[-\frac{2\pi}{h}\int_0^r \nu(r)\,dr\right] \qquad (8.11)$$

The factor 2π in (8.11) is replaced by π in boundary value problem 2°.

Let us note that the reasoning presented extends word for word to a system for which $E^{\alpha\beta} = A^{\alpha\beta}(x^\gamma) f(x)$, and E is an arbitrary positive function of x, x^α. This permits making a deduction about the exponential damping of the solutions in domains whose mapping on a cylinder results in an equation with the mentioned dependence of the coefficients on the coordinates.

Let us turn to an analysis of analogous questions for the system of equations of elasticity theory by considering that the elastic properties are invariant for reflection relative to a plane parallel to the middle plane

$$2U = E^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} + 2E^{\alpha\beta}\varepsilon_{\alpha\beta}\varepsilon + E\varepsilon^2 + 4G^{\alpha\beta}\varepsilon_\alpha\varepsilon_\beta, \quad \varepsilon_{0\alpha} \equiv \varepsilon_\alpha, \quad \varepsilon_{00} \equiv \varepsilon \qquad (8.12)$$

It turns out that, in contrast to the system (6.1), a penetrating solution (not decreasing exponentially) can exist even if the total forces and moments equal zero on the endface of each strip R (see Sect. 10). Nevertheless, a number of particular cases is successfully isolated when the energy damps exponentially. These cases are listed below.

1°. The coefficients are $E^{\alpha\beta} = 0$ (the Poisson's ratio equals zero in the isotropic case), T_+ and T_- are load-free $(p^{io} = 0)$, $p^{i\alpha}\nu_\alpha = q^i(x^\alpha, x)$ on $\Gamma_p \times [-h/2, h/2]$; $w_i = s_i(x^\alpha, x)$, on $\Gamma_w \times [-h/2, h/2]$; q^α and r^α are even functions of x, r° and q° are odd functions of x, and moreover

$$\langle q\rangle = 0 \text{ on } \Gamma_p, \quad \langle s\rangle = 0 \text{ on } \Gamma_w \qquad (8.13)$$

2°. $w^i = 0$ on T_+, either $w^i = 0$ or $p^{io} = 0$ on T_-, and the conditions are arbitrary on Ω.

3°. The coefficients $E^{\alpha\beta} = 0$, $w^\circ = 0$, $p^{\alpha\circ} = 0$ on T_+, $p^{io} = 0$ on T_-, the boundary conditions on Ω are the same as in problem 1, however, q^i and s^i are not absolutely subject to the evenness conditions.

The proof is carried out according to the same scheme as for the system (6.1).

9. On the construction of solutions of problems in thin cylindrical domains with an error less than any degree of relative thickness. The solution in thin cylindrical domains is ordinarily sought in the form of series in a small parameter $h/L = h_*$. The error in the approximate solutions obtained in this manner is $O(h_*^n)$, the exponent n depends on the number of terms retained in the series [15 — 17]. The assertions formulated in Sect. 8 permit an indication of the method of constructing the approximate solutions, which differ from the exact solution by a quantity less than any power of h_*.

Let us describe it in the example of the following boundary value problem for the system (6.1)

$$E w_{,x} = q_{\pm}(x^{\alpha}) \text{ on } T_+, T_-$$
$$E^{\alpha\beta} w_{,\alpha} \nu_{\beta} = q(x^{\alpha}, x) \text{ on } \Gamma_p \times [-h/2, h/2]$$
$$w = s(x^{\alpha}, x) \text{ on } \Gamma_w \times [-h/2, h/2]$$

We shall seek the solution in the form of a sum $w = w^*(x, x^{\alpha}) + u(x^{\alpha}) + w'(x, x^{\alpha})$, where w^* is the solution of the problem in an infinite layer bounded by the planes T_+ and T_-, the boundary values $q_{\pm}(x^{\alpha})$ are continued to all values of x^{α} by some method, $u(x^{\alpha})$ is the solution of the problem $(E^{\alpha\beta}u_{,\alpha})_{,\beta} = 0$ in T, $E^{\alpha\beta}u_{,\alpha}\nu_{\beta} = \langle q - E^{\alpha\beta}w^*_{,\alpha}\nu_{\beta}\rangle$ on Γ_p, and $u = \langle s - w^*\rangle$ on Γ_w. We therefore obtain the boundary value problem 1° for w'. Therefore, w' damps exponentially with distance from the edge with an exponent c/h in the exponential, hence, the error of the approximate solution $w = w^* + u$ as $h_* \to 0$ is less than any power of h_* at any fixed point. The continuation of the function $q_{\pm}(x^{\alpha})$ should be chosen so that the edge data for w' would be on the order of $q_{\pm}(x^{\alpha})$ or less. Then the amplitude of the exponentially decreasing function w' will be of the same order or less than the amplitude of the approximate solution.

The approximate solution of other boundary value problems is constructed analogously. Let us note that the property mentioned in Sect. 7 for the boundary value problem 3° for an elastic body permits the construction of a "hyperfine" solution of the problem concerning a stamp.

10. Supplement. Let us consider a plate whose face surfaces are load-free. Surface forces are given at the plate edge so that the total force and moment equal zero along each transverse fiber.

We shall show that the state of stress can nevertheless be penetrating, i.e., does not damp out exponentially with the velocity $\mathrm{const}\ h^{-1}$. We shall consider the plate semi-infinite $(|x| \leqslant h/2, |y| < +\infty, 0 \leqslant z \leqslant +\infty)$, isotropic, and for simplicity we set the Lamé coefficient λ equal to zero. We take the boundary conditions in the form

$$\sigma_{zz} = 2\mu k \cos ky \left(x + \sum_{s=1}^{3} a_s \varkappa_s \sin(2s-1)\pi x h^{-1}\right) \tag{10.1}$$

$$\sigma_{yz} = 2\mu \sin ky \left(kx + \frac{1}{2}\sum_{s=1}^{3} a_s (\varkappa_s^2 + k^2)\sin(2s-1)\pi x h^{-1}\right)$$

$$\sigma_{xz} = -\mu k h^{-1}\pi \cos ky \sum_{s=1}^{3} a_s (2s-1)\cos(2s-1)\pi x h^{-1}$$

$$\varkappa_s = h^{-1}(\pi^2(2s-1)^2 + k^2h^2)^{1/2}$$

Here μ is the shear modulus, and k, a_s are fixed numbers. It is easy to construct the solution of this problem

$$w_x = -k^{-1}e^{-kz}\cos ky \tag{10.2}$$
$$w_y = -xe^{-kz}\sin ky - \sin ky\sum_{s=1}^{3} a_s\varkappa_s e^{-\varkappa_s z}\sin(2s-1)\pi x h^{-1}$$
$$w_z = -xe^{-kz}\cos ky - k\cos ky\sum_{s=1}^{3} a_s e^{-\varkappa_s z}\sin(2s-1)\pi x h^{-1}$$

The solution has been constructed from biharmonic and vortex solutions [15]. It contains a penetrating part (the first terms in (10.2)) and a boundary layer.

Let us impose constraints on the boundary data (10.1): $\langle\sigma_{zz}x\rangle = \langle\sigma_{yz}x\rangle = \langle\sigma_{xz}\rangle = 0$. These constraints are a system of three linear inhomogeneous equations in a_1, a_2, a_3. The system determinant Δ is given by the formula $\Lambda = \mu^3k^4h^4(2\varkappa_1 - 3\varkappa_2 + 4\varkappa_3)$ and differs from zero. Let us take the solution of this system as a_s in (10.1), then the total force and moment on each transverse fiber will equal zero for $z = 0$.

It can be verified that the a_s are on the order of h^2, therefore, σ_{yz}, $\sigma_{yx} \sim 1$, σ_{zz}, $\sigma_{xz}, \sigma_{yy} \sim h$ for $z = 0$. In the penetrating part of the solution σ_{zz}, σ_{yz}, $\sigma_{yy} \sim h$, $\sigma_{xx} = \sigma_{xz} = \sigma_{xy} \equiv 0$. This is in agreement with the formulation of the Saint — Venant principle in the theory of plates given by Vorovich [15].

The example constructed shows that the refined two-dimensional equations of plate theory which takes account only the total force and moment on the edge, cannot correctly describe corrections on the order of h. It also follows from this example that when only the total force and moment are known on each transverse fiber at the plate edge, but the exact surface force distribution along the transverse fibers is unknown, the construction of the state of stress with terms on the order of h taken into account is impossible, and the greatest achievable accuracy is yielded by the classical two-dimensional equations.

REFERENCES

1. De Saint — Venant, B., Memoire sur la torsion des prismes. Mem. Divers Savants, Vol. 14, 1855.

2. Boussinesq, M.S., Application des Potentiels a l' Etude de l' Equilibre et du Mouvement des Solides Elastiques. Gauthier — Villars, Paris, 1885. Nouveau tirage, 1969.

3. Von Mises, R., On Saint — Venant's principle, Bull. Amer. Math. Soc., 51, No. 8, 1945.

4. Sternberg, E., On Saint — Venant's principle, Quart. Appl. Math. Vol. No. 4, 1954.

5. Toupin, R.A., Saint — Venant's principle. Arch. Ration. Mech. and Analysis, Vol. 18, No. 2, 1965.

6. Berdichevskii, V.L., On the proof of the Saint — Venant principle for bodies of arbitrary shape. PMM, Vol. 38, No. 5, 1974.

7. Diaz, J.B. and Payne, L.E., Mean value theorems in the theory of elasticity, Proc. Third US Nat. Congr. of Appl. Mech., Providence, 1958, p. 293, ASME, New York, 1958.

8. Oleinik, O.A. and Iosif'ian, G. A., Energy estimates of generalized solutions of boundary value problems for second order elliptic equations and their application, Dokl. Akad. Nauk SSSR, Vol. 232, No.6, 1977.

9. Oleinik, O.A. and Iosif'ian, G.A., Saint — Venant's principle for the mixed problem of elasticity theory and its application, Dokl. Akad. Nauk SSSR, Vol. 232, No.5, 1977.

10. Oleinik, O.A. and Yosifian, G.A., Boundary value problems for second order elliptic equations in unbounded domains and Saint — Venant's principle. Ann. Scuola Norm. Super. Pisa. Sci. Fis. e Mat., Ser. 4, Vol. 4, No. 2, 1977.

11. Oleinik, O.A. and Yosifian, G.A., On singularities at the boundary points and uniqueness theorems for solutions of the first boundary value problems of elasticity. Comm. Partial Differential Equations. Vol. 2, No.9, 1977.

12. Oleinik, O.A. and Iosif'ian, G.A., Analogue of the Saint— Venant principle and uniqueness of the solutions of boundary value problems in unbounded domains for parabolic equations in unbounded domains. Usp. Matem. Nauk, Vol. 31, No.6, 1976.

13. Sedov, L. I., Similarity and Dimensional Analysis Methods in Mechanics. "Nauka", Moscow, 1977.

14. Mikhlin, S.G., Variational Methods in Mathematical Physics. (English translation). Pergamon Press, Book No. 10146, 1964.

15. Vorovich, I. I., Some results and problems of the asymptotic theory of plates and shells, Materials, First All-Union School on the Theory and Numerical Methods of Shell Analysis. (Gegechkori, GruzSSR, 1974), Isz, Tbilisi Univ. 1975.

16. Gol'denveizer, A.L., Asymptotic method of constructing a theory of shells. Materials, All-Union School on the Theory of Shells and Numerical Methods of Analysis. (Gegechkori, GruzSSR, 1974), Izd. Tbilisi Univ, 1975.

17. Dzhavadov, M.G., Asymptotics of the solution of a boundary value problem for second order elliptic equations in thin domains, Differents. Uravneniia, (Differential Equations), Vol. 4, No.10, 1968.

Translated by M. D. F.